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Estimations of Domains with Cycles

A. P. KRISHCHENKO
Department of Applied Mathematics, N.E. Bauman Moscow State Technical University
ul. 2-aja Baumanskaja 5, 107005 Moscow, Russia
nonlin@cs.msu.su

Abstract—A method for estimating domains with limit cycles and finding surfaces with the traces of all cycles is proposed. Corresponding estimations of domains with cycles for Lorenz and Rössler systems are indicated.

Keywords—Cycle, Vector field, Lie derivative.

1. INTRODUCTION

Many properties of nonlinear systems of the form

$$\dot{x} = f(x), \quad x = (x_1, \dots, x_n)^\top \in \mathbf{R}^n, \quad f(x) = (f_1(x), \dots, f_n(x))^\top \in C^\infty(\mathbf{R}^n) \tag{1.1}$$

depend on the existence of cycles in its phase space $\mathbf{R}^n$. This means that for qualitative description of properties of the system (1.1), it is always important to localize the cycles, i.e., to solve the following problems.

PROBLEM 1. Find such set $\Omega \subset \mathbf{R}^n$ which contains all cycles of the system (1.1).

PROBLEM 2. Find such surface $S \subset \mathbf{R}^n$ which intersects or touches any cycle of the system (1.1).

We consider these localization problems for cycles corresponding to periodic solutions $x(t)$, $x(.) \in C^2(\mathbf{R})$ of system (1.1) with finite period.

This paper is organized as follows. The main results are formulated in Section 2 and are proved in Section 3. Sections 4–6 contain results received for the Lorenz [1] and Rössler systems [2]. In Section 6, an iteration procedure is proposed for localization of cycles.

2. GENERAL METHOD

Let

$$f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$$

be a vector field on $\mathbf{R}^n$ associated with system (1.1), and let

$$L_f \varphi(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial \varphi(x)}{\partial x_i}$$

be a Lie derivative of $\varphi \in C^\infty(\mathbf{R}^n)$ along the vector field f.

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THEOREM 1. *For any function $\varphi \in C^\infty(\mathbf{R}^n)$, each cycle of system (1.1) has at least two points contained in the set*

$$S_\varphi = \{x \mid L_f\varphi(x) = 0\}. \tag{2.1}$$

THEOREM 2. *For any function $\varphi \in C^\infty(\mathbf{R}^n)$, each cycle of system (1.1) has at least one point contained in the set*

$$S_\varphi^- = \{x \in S_\varphi \mid L_f^2\varphi(x) \le 0\}, \tag{2.2}$$

and at least one point contained in the set

$$S_\varphi^+ = \{x \in S_\varphi \mid L_f^2\varphi(x) \ge 0\}. \tag{2.3}$$

THEOREM 3. *Any cycle of system (1.1) intersects the set S_φ transversely at each point in common with the set*

$$\{x \in S_\varphi \mid L_f^2\varphi < 0\}, \tag{2.4}$$

and with the set

$$\{x \in S_\varphi \mid L_f^2\varphi > 0\}, \tag{2.5}$$

where $\varphi \in C^\infty(\mathbf{R}^n)$.

The sets (2.4), (2.5) are $(n-1)$-dimensional submanifolds (hypersurfaces) in $\mathbf{R}^n$.

For function $\varphi \in C^\infty(\mathbf{R}^n)$, by definition put

$$\varphi_{\sup} = \sup\{\varphi(x) \mid L_f\varphi(x) = 0\}, \qquad \varphi_{\inf} = \inf\{\varphi(x) \mid L_f\varphi(x) = 0\}. \tag{2.6}$$

THEOREM 4. *All cycles of system (1.1) are contained in*

$$\Omega_\varphi = \{x \mid \varphi_{\inf} \le \varphi(x) \le \varphi_{\sup}\}. \tag{2.7}$$

COROLLARY 1. *All cycles of system (1.1) are contained in*

$$\Omega = \{\cap\Omega_\varphi \mid \varphi \in C^2(\mathbf{R}^n)\}.$$

COROLLARY 2. *For any functions $\varphi, \psi \in C^\infty(\mathbf{R}^n)$, any cycle of system (1.1) has at least one point in common with the set $S_\psi^- \cap \Omega_\varphi$ and with the set $S_\psi^+ \cap \Omega_\varphi$.*

If $S_\varphi = \emptyset$ for some function φ, or $S_\psi^- \cap \Omega_\varphi = \emptyset$ ($S_\psi^+ \cap \Omega_\varphi = \emptyset$) for some two functions φ and ψ, then system (1.1) has no cycles. If S_φ consists of a finite number of points, then it is enough to investigate behavior of trajectories passing through these points.

For example, suppose x_* is an asymptotically stable point on the whole equilibrium of system (1.1), and $\varphi = V(x)$ is the corresponding Lyapunov function. In this case, we get $S_\varphi = x_*$.

Suppose S_φ is a compact set, then Ω_φ is limited by the surfaces $\varphi(x) = \varphi_{\inf}$ and $\varphi(x) = \varphi_{\sup}$.

For the estimation problem, a boundedness property of the sets $\{x \mid L_f\varphi(x) = 0\}$ and $\{x \mid \varphi(x) = \text{const}\}$ is important. Suppose system (1.1) is a polynomial system, i.e., the functions $f_i(x)$, $i = 1, \dots, n$ are polynomials, and $\varphi(x)$ is also a polynomial. In this case, function $L_f\varphi(x)$ is a polynomial, too. If we combine this with the boundedness conditions of these sets, we get some conditions on the coefficients of the polynomial $\varphi(x)$. In the case of a polynomial of degree 2, the terms of second degree compose a quadratic form, and we have simplest boundedness criterion for such polynomial: its quadratic form is negative (or positive) definite.

To formulate the boundedness conditions for a polynomial of degree $n > 2$, let us introduce the following notation. Let $P_m(x)$ be a polynomial of degree m,

$$P_m(x) = \sum_{|\sigma|=0}^{m} a_\sigma x^\sigma, \tag{2.8}$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$, σ_i are whole nonnegative numbers,

$$|\sigma| = \sum_{i=1}^{n} \sigma_i, \qquad x^\sigma = x_1^{\sigma_1} x_2^{\sigma_2} \dots x_n^{\sigma_n},$$

$a_\sigma \in \mathbf{R}$, $a_\sigma \neq 0$ for some σ $|\sigma| = m$, and

$$\pi = \{x \in \mathbf{R}^n \mid P_m(x) = 0\}. \tag{2.9}$$

PROPOSITION 1. *If set (2.9) is bounded, for any $1 \le i_1 < i_2 < \dots < i_k \le n$, polynomial $P_m(x)$ (2.8) is a polynomial of even degree as a function of the variables $x_{i_1}, x_{i_2}, \dots, x_{i_k}$.*

In what follows, we suppose $m = 2l$. If $x = t\beta$, where $\beta = \{\beta_1, \dots, \beta_n\}$, $\|\beta\|^2 = \sum_{i=1}^n \beta_i^2 = 1$, $t > 0$, then

$$P_m(t\beta) = \sum_{k=0}^{m} A_k(\beta) t^k, \qquad \text{where } A_k(\beta) = \sum_{|\sigma|=k} a_\sigma \beta^\sigma, \quad k = 0, \dots, m.$$

We put

$$A_m = \min\{|A_m(\beta)| \text{ for } \|\beta\| = 1\}.$$

PROPOSITION 2. *If $A_m > 0$, then set (2.9) is bounded.*

PROPOSITION 3. *If $A_m(\beta_*) = 0$, and $A_{m-1}(\beta_*) \neq 0$ or*

$$\operatorname{rank} \left. \frac{\partial (A_m(\beta), \|\beta\|^2)^\top}{\partial \beta} \right|_{\beta=\beta_*} = 2,$$

then set (2.9) is unbounded.

PROPOSITION 4. *If for all $\beta, \|\beta\| = 1$*

$$A_m(\beta) \ge 0 (\le 0), \qquad A_{m-1}(\beta) \equiv \dots \equiv A_{m-2k+1}(\beta) \equiv 0,$$

and $A_{m-2k}(\beta)|_{A_m(\beta)=0} > 0$ ($A_{m-2k}(\beta)|_{A_m(\beta)=0} < 0$), then set (2.9) is bounded.

3. PROOF OF THEOREMS 1–4

To prove Theorems 1–4, we need the following facts. Let $x(t) \in C^2(\mathbf{R}^n)$ be a T-periodic solution of system (1.1), and let $\varphi(x)$ be a function from $C^2(\mathbf{R}^n)$.

The function $\varphi(x(t))$ is a periodic function of t and we can consider $\varphi(x(t))$ as a function onto a circle. A circle is a compact set. Hence, there exist a maximum and minimum, i.e., there exist $t = t^*$ and $t = t_*$ for which

$$\varphi(x(t_*)) \le \varphi(x(t)) \le \varphi(x(t^*)). \tag{3.1}$$

Using the necessary condition of local extremum, we get

$$L_f \varphi(x)|_{x=x(t)} = \frac{d\varphi(x(t))}{dt} = 0, \tag{3.2}$$

where $t = t^*$ or $t = t_*$. It is obvious that condition (3.2) can be fulfilled only at the points of the set $L_f \varphi(x) = 0$. Theorem 1 is proved.

At the point of maximum, we have

$$L_f^2 \varphi(x)|_{x=x(t)} = \frac{d^2\varphi(x(t))}{dt^2} \le 0. \tag{3.3}$$

Condition (3.3) can be fulfilled only at the points of set (2.2). Similarly, $x(t_*) \in S_\varphi^+$. This completes the proof of Theorem 2.

The Lie derivative $L_f^2 \varphi(x)$ can be interpreted as the scalar product of vectors $f(x)$ and $\operatorname{grad} L_f \varphi(x)$. Therefore, if $L_f^2 \varphi(x) \neq 0$, then $\operatorname{grad} L_f \varphi(x) \neq 0$ and trajectory $x(t)$ intersects surface $L_f \varphi(x) = 0$ transversely. This proves Theorem 3.

The proof of Theorem 4 follows from inequalities

$$\varphi_{\inf} \le \min\{\varphi(x(t))\} \le \max\{\varphi(x(t))\} \le \varphi_{\sup},$$

because any cycle cannot pass through the point in which $\varphi(x) < \varphi_{\inf}$ or $\varphi_{\sup} < \varphi(x)$.

The proofs of Corollaries 1 and 2 are trivial.

4. THE LORENZ SYSTEM

The Lorenz system has the form

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - b x_3, \end{aligned} \tag{4.1}$$

where σ, r, and b are positive parameters.

PROPOSITION 5. *For a polynomial of degree 2 of the form*

$$\varphi(x) = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 + 2\delta x_1 x_2 + 2\varepsilon x_1 x_3 + 2\mu x_2 x_3 + 2\nu x_2 + 2\lambda x_3 + 2\theta x_1 + \tau, \tag{4.2}$$

the following three conditions:

1° $\deg L_f\varphi(x) = 2$,

2° *the terms of second degree of function $L_f\varphi(x)$ compose a negative definite quadratic form, and*

3° *the terms of second degree of function $\varphi(x)$ compose a positive definite quadratic form,*

are fulfilled iff

$$\alpha > 0, \quad \beta = \gamma > 0, \quad \delta = \varepsilon = \mu = 0, \tag{4.3}$$

$$|\alpha\sigma + \beta r + \lambda| < \sqrt{4\alpha\beta\sigma - \frac{\nu^3}{b}}, \tag{4.4}$$

i.e.,

$$\varphi(x) = \alpha x_1^2 + \beta x_2^2 + \beta x_3^2 + 2\nu x_2 + 2\lambda x_3 + 2\theta x_1 + \tau, \tag{4.5}$$

where $\alpha > 0$, $\beta > 0$, and inequality (4.4) holds.

If the conditions 1°–3° hold, the set Ω_φ is bounded by an ellipsoid.

PROOF. Since

$$\begin{aligned} L_f\varphi(x) = {} & 2\alpha\sigma x_1(-x_1 + x_2) + 2\beta x_2(r x_1 - x_2 - x_1 x_3) + 2\gamma x_3(x_1 x_2 - b x_3) \\ & + 2\delta x_1(r x_1 - x_2 - x_1 x_3) + 2\delta x_2 \sigma(-x_1 + x_2) \\ & + 2\varepsilon x_3 \sigma(-x_1 + x_2) + 2\varepsilon x_1(x_1 x_2 - b x_3) \\ & + 2\mu x_2(x_1 x_2 - b x_3) + 2\mu x_3(r x_1 - x_2 - x_1 x_3) \\ & + 2\nu(r x_1 - x_2 - x_1 x_3) + 2\lambda(x_1 x_2 - b x_3) + 2\theta\sigma(-x_1 + x_2), \qquad \text{i.e.,} \\ L_f\varphi(x) = {} & 2(\gamma - \beta) x_1 x_2 x_3 - 2\delta x_1^2 x_3 + 2\varepsilon x_1^2 x_2 + 2\mu x_1 x_2^2 - 2\mu x_1 x_3^2 + \cdots, \end{aligned}$$

then the first condition is fulfilled iff

$$\gamma = \beta, \qquad \delta = \varepsilon = \mu = 0.$$

Therefore the polynomial $\varphi(x)$ has the form (4.5) and

$$L_f\varphi(x) = -2\alpha\sigma x_1^2 - 2\beta x_2^2 - 2\beta b x_3^2 + (2\alpha\sigma + 2\beta r + 2\lambda) x_1 x_2 - 2\nu x_1 x_3 + \cdots.$$

Conditions 2° and 3° are fulfilled iff $\alpha > 0$, $\beta > 0$, $-2\alpha\sigma < 0$, and

$$\begin{aligned} 4\alpha\beta\sigma - (\alpha\sigma + \beta r + \lambda)^2 > 0, \\ -8\alpha\beta^2\sigma b + 2\nu^2\beta + 2\beta b(\alpha\sigma + \beta r + \lambda)^2 < 0. \end{aligned} \tag{4.6}$$

To conclude the proof, it remains to note that condition (4.4) coincides with (4.6). For

$$\alpha = r, \qquad \beta = \sigma, \qquad \lambda = -4r\sigma, \qquad \nu = 0, \qquad \tau = 4\sigma r^2,$$

we get the function [3]

$$\varphi = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2.$$

If we put

$$\alpha = \frac{1}{2\sigma}, \qquad \beta = \frac{1}{2}, \qquad \lambda = -\frac{r+1}{2}, \qquad \nu = 0, \qquad \tau = \frac{(r+1)^2}{2},$$

then

$$\varphi = \frac{1}{2\sigma}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}(x_3 - (r+1))^2. \tag{4.7}$$

For function (4.7)

$$L_f\varphi = -x_1^2 - x_2^2 - bx_3^2 + (r+1)bx_3,$$

and the surface S_φ is an ellipsoid

$$\frac{x_1^2}{A^2} + \frac{x_2^2}{B^2} + \frac{(x_3 - (r+1)/2)^2}{C^2} = 1, \qquad A = B = \sqrt{b}\,C, \quad C = \frac{(r+1)}{2}, \tag{4.8}$$

of center $O_1(0, 0, (r+1)/2)$,

$$\varphi_{\text{inf}} = 0, \qquad \varphi_{\text{sup}} = \rho,$$

where [4]

$$\rho = \rho(r, b, \sigma) = \begin{cases} \dfrac{(r+1)^2}{2}, & 0 < b \le 2, \quad \sigma \ge \dfrac{b}{2}, \\ (r+1)^2b^2/(8(b-1)), & 2 < b, \quad \sigma \ge 1, \\ (r+1)^2b^2/(8(b-\sigma)\sigma), & 0 < \sigma < 1, \quad b > 2\sigma. \end{cases} \tag{4.9}$$

The condition $\varphi_{\text{inf}} \le \varphi(x)$ is fulfilled for all x and does not impose the restrictions. On the other hand, condition $\varphi(x) \le \varphi_{\text{sup}}$ defines a set Ω_φ limited by the ellipsoid

$$\frac{x_1^2}{A^2} + \frac{x_2^2}{B^2} + \frac{(x_3 - (r+1))^2}{C^2} = 1, \qquad A = \sqrt{\sigma}B, \quad B = C = \sqrt{2\rho}, \tag{4.10}$$

of center $O_2(0, 0, r+1)$.

All cycles of system (4.1) place inside ellipsoid (4.10). All trajectories from the external set (relatively ellipsoid (4.10)) fall into ellipsoid (4.10) and remain there. Function (4.7) decreases on trajectories of system (4.1) outside of ellipsoid (4.8), and it grows inside ellipsoid (4.8). The chaotic movement in system (4.1) is possible only inside ellipsoid (4.10).

By Proposition 5, it follows that for any function (4.5), we have the similar qualitative results.

5. THE RÖSSLER SYSTEM

For the Rössler system

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + ax_2, \\ \dot{x}_3 &= c + x_3(x_1 - b), \end{aligned} \tag{5.1}$$

where a, b, and c are positive parameters, the method allows to find the function $\varphi = x_1^2 + x_2^2 + 2x_3$. For this function, surface S_φ is a parabolic cylinder

$$ax_2^2 - bx_3 + c = 0,$$

and $\varphi_{\text{inf}} = 2c/b$, $\varphi_{\text{sup}} = +\infty$. All cycles of system (5.1) place outside of paraboloid

$$x_3 = \frac{c}{b} - \frac{(x_1^2 + x_2^2)}{2}. \tag{5.2}$$

For function $\psi(x) = x_2$, $S_\psi = \{L_f\psi = 0\} = \{x \mid x_1 + ax_2 = 0\}$ and according to Corollary 2 and Theorem 2, each cycle of system (5.1) has at least one point in common with the part of plane $x_1 + ax_2 = 0$ which is outside of the domain restricted by surface (5.2) and where $ax_3 \ge x_1$. If $ax_3 > x_1$, then this point is a point of intersection in transverse position.

6. ITERATION PROCEDURE OF LOCALIZATION

THEOREM 5. *If all cycles of system (1.1) are contained in* $\Lambda \subset \mathbf{R}^n$, *then for any function* $\chi \in C^2(\mathbf{R}^n)$, *the sets*

$$\Omega_{\chi,\Lambda} = \{x \mid \chi_{\inf,\Lambda} \le \chi(x) \le \chi_{\sup,\Lambda}\}, \tag{6.1}$$

$$\Lambda_\chi = \Omega_{\chi,\Lambda} \cap \Lambda \tag{6.2}$$

contain also all cycles. In (6.1), (6.2)

$$\chi_{\inf,\Lambda} = \inf\{\chi(x) \mid x \in S_\chi \cap \Lambda\}, \qquad \chi_{\sup,\Lambda} = \sup\{\chi(x) \mid x \in S_\chi \cap \Lambda\}, \tag{6.3}$$

where $S_\chi = \{x \mid L_f\chi(x) = 0\}$.

PROOF. If $x(t)$ is a periodic solution of system (1.1), and t^*, t_* denote a maximum and a minimum of function $\chi(x(t))$, then $x(t_*), x(t^*) \in S_\chi \cap \Lambda$ and

$$\chi(x(t)) \le \chi(x(t^*)) \le \sup\{\chi|_{S_\chi\cap\Lambda}\} = \chi_{\sup,\Lambda}.$$

Inequality $\chi(x(t)) \ge \chi_{\inf,\Lambda}$ can be proved analogously. Therefore, for all t we have $x(t) \in \Omega_{\chi,\Lambda}$.

COROLLARY 3. *Let* $\chi_n(x)$, $n = 0, 1, 2, \dots$, *be a sequence of functions from* $C^2(\mathbf{R}^n)$. *The sets*

$$\Lambda_0 = \Omega_{\chi_0}, \quad \Lambda_n = \Lambda_{n-1} \cap \Lambda_{n-1,\chi_n}, \qquad n > 0, \tag{6.4}$$

contain all cycles of system (1.1) and

$$\Lambda_0 \supseteq \Lambda_1 \supseteq \cdots \supseteq \Lambda_n \supseteq \cdots. \tag{6.5}$$

COROLLARY 4. *Let* $x_i(t)$, $i = 1, 2, 3$, *be a periodic solution of Lorenz system (4.1). Then the following estimates hold for any* t:

$$|x_1(t)| \le \sqrt{\frac{2\rho\sigma}{\sigma+1}} \triangleq x_{1*}, \tag{6.6}$$

where ρ *is from (4.9), and*

$$\begin{aligned} x_3(t) &\ge \frac{x_1^2(t)}{2\sigma} \ge 0, && b < 2\sigma, \\ x_3(t) &= \frac{x_1^2(t)}{2\sigma}, && b = 2\sigma, \\ x_3(t) &\le \frac{x_1^2(t)}{2\sigma}, && b > 2\sigma. \end{aligned} \tag{6.7}$$

PROOF. If $\chi_0(x) = \varphi(x)$ from (4.7), $\chi_1(x) = \psi(x) = x_1$, by Corollary 3, it follows that the sets

$$\Lambda_0 = \Omega_{\chi_0} = \Omega_\varphi, \qquad \Lambda_1 = \Lambda_0 \cap \Lambda_{0,\chi_1} = \Omega_\varphi \cap \Omega_{\varphi,\psi}$$

contain all cycles of system (4.1). Therefore,

$$\inf\{\psi(x) \mid x \in S_\psi \cap \Omega_\varphi\} \le \psi(x(t)) = x_1(t) \le \sup\{\psi(x) \mid x \in S_\psi \cap \Omega_\varphi\}.$$

To find $\inf\{\psi(x) \mid x \in S_\psi \cap \Omega_\varphi\}$ and $\sup\{\psi(x) \mid x \in S_\psi \cap \Omega_\varphi\}$, we consider

$$x_1 \to \text{extr}, \qquad -x_1 + x_2 = 0, \qquad \frac{1}{\sigma}x_1^2 + x_2^2 + (x_3 - (r+1))^2 \le 2\rho. \tag{6.8}$$

If we replace x_2 by x_1 in (6.8), we obtain

$$x_1 \to \text{extr}, \qquad \left(\frac{1}{\sigma}+1\right)x_1^2 + (x_3-(r+1))^2 \le 2\rho.$$

Therefore

$$x_1^2 \le \frac{2\rho}{1/\sigma+1},$$

and we obtain estimate (6.6). To prove (6.7), we need the function [4]

$$\chi(x) = x_1^2 - 2\sigma x_3. \tag{6.9}$$

For this function, $L_f\chi(x) = 2x_1\sigma(-x_1+x_2) - 2\sigma(x_1x_2 - bx_3) = 2\sigma(bx_3 - x_1^2)$ and

$$S_\chi = \{x \mid bx_3 - x_1^2 = 0\}.$$

Therefore

$$\chi(x)|_{S_\chi} = x_1^2\left(1-\frac{2\sigma}{b}\right). \tag{6.10}$$

Let us consider three cases.

CASE 1. $b < 2\sigma$. From (6.10) it follows that $\chi(x)|_{S_\chi} \le 0$, $\chi_{\sup} = 0$, $\chi_{\inf} = -\infty$. From Theorem 4 it follows that all cycles of system (4.1) are contained in

$$\Omega_\chi = \{x \mid \chi(x) \le 0\} = \{x \mid x_1^2 - 2\sigma x_3 \le 0\}. \tag{6.11}$$

Therefore for all cycles $x_3 \ge x_1^2/(2\sigma) \ge 0$.

CASE 2. $b > 2\sigma$. From (6.10) it follows that $\chi(x)|_{S_\chi} \ge 0$, $\chi_{\sup} = +\infty$, $\chi_{\inf} = 0$. From Theorem 4 it follows that all cycles of system (4.1) are contained in

$$\Omega_\chi = \{x \mid \chi(x) \ge 0\} = \{x \mid x_1^2 - 2\sigma x_3 \ge 0\}.$$

Therefore for all cycles $x_3 \le x_1^2/(2\sigma)$.

CASE 3. $b = 2\sigma$. From (6.10) it follows that $\chi(x)|_{S_\chi} = 0$, $\chi_{\sup} = \chi_{\inf} = 0$. From Theorem 4 it follows that all cycles of system (4.1) are contained in

$$\Omega_\chi = \{x \mid \chi(x) = 0\} = \{x \mid x_1^2 - 2\sigma x_3 = 0\} = S_\chi.$$

Therefore for all cycles $x_3 = x_1^2/(2\sigma)$. This completes the proof.

REMARK 1. In Case 3, any cycle of system (4.1) intersects the set $\Omega_\varphi \cap \Omega_\chi \cap S_\psi$, where the set Ω_φ is limited by ellipsoid (4.10), and $S_\psi = \{x \mid x_1 = x_2\}$. This set is described by the system

$$\frac{1}{\sigma}x_1^2 + x_2^2 + (x_3-(r+1))^2 \le 2\rho, \qquad bx_3 - x_1^2 = 0, \qquad x_1 - x_2 = 0.$$

Hence,

$$x_1 = \tau, \qquad x_2 = \tau, \qquad x_3 = \frac{\tau^2}{b}, \tag{6.12}$$

where

$$\frac{\tau^4}{b^2} + \tau^2\left(1-\frac{2r}{b}\right) + ((r+1)^2 - 2\rho) \le 0.$$

If $0 < b \le 2$ ($b = 2\sigma$), from (4.9) we obtain $\rho = (r+1)^2/2$. Therefore, under $1 - 2r/b \ge 0$, i.e., $0 < r \le b/2$, we get $\tau = 0$, $x = 0$ and system (4.1) has no cycles. But if $1 - 2r/b < 0$, i.e., $r > b/2$, then $|\tau| \le \sqrt{k}$, where $k = -b\sqrt{1-2r/b}$.

If $b = 2\sigma > 2$, we have $\rho = (r+1)^2 b^2/(8(b-1))$, $(r+1)^2 - 2\rho = -(r+1)^2(b-2)^2/(4(b-1)) < 0$. Therefore, $|\tau| \leq \sqrt{k}$, where k is the positive root of equation

$$\frac{k^2}{b^2} + k\left(1 - \frac{2r}{b}\right) - \frac{(r+1)^2(b-2)^2}{4(b-1)} = 0.$$

Equalities (6.12) define possible initial conditions of cycles for all these obtained values of τ.

REMARK 2. In Case 2, denote by Λ the set $\Omega_\varphi \cap \Omega_\chi$, where the set Ω_φ is limited by ellipsoid (4.10), and the set Ω_χ coincides with (6.11). It follows from Theorem 5 that all cycles are contained in

$$\Omega_{\psi,\Lambda} = \{x \mid \psi_{\mathrm{inf},\Lambda} \leq \psi(x) \leq \psi_{\mathrm{sup},\Lambda}\}.$$

Let ψ be the function x_1. To determine $\psi_{\mathrm{inf},\Lambda}, \psi_{\mathrm{sup},\Lambda}$, we need so solve the following problem:

$$x_1 \to \mathrm{extr}, \qquad \frac{1}{\sigma}x_1^2 + x_2^2 + (x_3 - (r+1))^2 \leq 2\rho, \qquad -x_1 + x_2 = 0, \qquad x_1^2 - 2\sigma x_3 \leq 0. \quad (6.13)$$

Taking into account (4.9) and the proof of (6.6) in the case

$$2 \leq b \leq 2\sigma, \tag{6.14}$$

for

$$x_3 = r + 1, \qquad |x_1| = \sqrt{\frac{2\rho\sigma}{\sigma+1}},$$

we obtain

$$\begin{aligned} x_1^2 - 2\sigma x_3 &= \frac{2\rho\sigma}{\sigma+1} - 2\sigma(r+1) \\ &= \frac{\sigma(r+1)^2 b^2}{4(b-1)(\sigma+1)} - 2\sigma(r+1) \\ &= \sigma(r+1)\left(\frac{(r+1)b^2}{4(b-1)(\sigma+1)} - 2\right) \geq 0, \end{aligned}$$

under

$$r \geq \frac{8(b-1)(\sigma+1)}{b^2} - 1. \tag{6.15}$$

Therefore, the maximum and minimum of x_1 in (6.13) will occur for (6.14), (6.15) somewhere on the set $\{x_1^2 - 2\sigma x_3 = 0\}$. Hence, for (6.14), (6.15) we get

$$\begin{aligned} &x_1 \to \mathrm{extr}, && \frac{1}{\sigma}x_1^2 + x_1^2 + (x_3 - (r+1))^2 = 2\rho, \qquad x_1^2 - 2\sigma x_3 = 0, \qquad \text{or} \\ &x_3 \to \max, && 2(1+\sigma)x_3 + (x_3 - (r+1))^2 = 2\rho. \end{aligned}$$

Thus in the case (6.14), (6.15) we have

$$x_{3,\max} = (r - \sigma) + \sqrt{(r-\sigma)^2 - (r+1)^2 + 2\rho},$$

where $\rho = (r+1)^2 b^2/(8(b-1))$.

Hence, if $2 \leq b \leq 2\sigma$, $r \geq 8(b-1)/b^2 - 1$, for any cycle we get

$$|x_1(t)| \leq \sqrt{2\sigma x_{3,\max}} \triangleq x_{1,\max}. \tag{6.16}$$

REMARK 3. In the canonical case ($\sigma = 10$, $b = 8/3$, $r = 28$) using (6.6) and (4.10), we get numerical estimate for domain with all cycles of system (4.1)

$$|x_1| < 28.56, \qquad |x_2| < 29.96, \qquad 0 < x_3 < 58.96.$$

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